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DECOMPOSITION OF THE STATIONARY ISOTROPIC TRANSPORT OPERATOR IN THREE INDEPENDENT SPACE VARIABLES

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Applied Mathematics Division

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ABSTRACT

Based on the idea of separation of variables, a decomposition of the three-dimensional linear transport operator is carried out, resulting in continuous sets of regular and generalized eigenfunctions. Because of the nonself-adjoint nature of this operator, the results could not be anticipated intuitively from the known decomposition of this operator in the special case of plane geometry. The results obtained may indicate a new approach to spectral theory.

1. INTRODUCTION

The decomposition of a linear transport operator for plane geometry has been successfully attempted in several different ways (see References 1 to 4). Whether explicitly expressed in these papers or not, each one tried to build a spectral theory for this operator.

In this report, a decomposition of the transport operator in three space-dimensions is carried out. The general idea is that of References 2 and 3, for the author was unable to generalize his ideas in References 1. The reason, obviously, is that the transport operator is not self-adjoint, and therefore no general spectral theory exists for it, as of today. The results of the expansion theorem proved rather surprising and could not be anticipated by intuition. On the other hand, the known expansion theorem for plane geometry with its much simpler structure is, of course, a special case of the general theory. The method used in this paper is suggested by the principle of separation of variables, which is widely used for the decomposition of linear partial differential equations of mathematical physics. To attempt completeness, it was necessary to add to the set of regular eigenfunctions, $\psi^0_\Lambda(\Omega)$, over a complex continuum, a set of generalized eigenfunctions, $\psi_\Lambda(\Omega)$, over a complex continuum of even higher dimension.

The regular eigenfunctions are not characterized by an eigenvalue, but by a complex vector which we may call an eigenvalue vector Λ. These eigenvalue vectors, however, have to satisfy the condition $\Lambda \cdot \Lambda = \lambda_0^2$, where

λ_0^2 assumes exactly one value for any given equation, and therefore $\pm\lambda_0$ can be called eigenvalues. To obtain the generalized eigenfunctions, it is necessary to evaluate the improper integral which occurs in Eq. (2.13) below. Its regular form as it appears in (2.9) has been known for a long time, but its improper form and its interpretation required careful analysis in order to formulate the decomposition theorem.

We turn now to the description of the operator. Given is the stationary transport equation,

$$v\Omega \cdot \nabla\psi + v\sigma\psi = \frac{c}{4\pi} v\sigma \int_{\{\Omega'\}} f(\Omega \cdot \Omega')\psi(r,\Omega')d_2\Omega' + Q(r,\Omega,v), \tag{1.1}$$

where

- $\psi = \psi(r,\Omega)$ is the directional flux, a scalar function;
- r is the position vector;
- v is a constant velocity in the direction of the unit vector Ω;
- σ is the total macroscopic cross section at velocity v;
- c is the net number of neutrons produced per collision $\left(=\dfrac{\sigma_s+\nu\sigma_f}{\sigma}\right)$;
- f is the scattering function;
- Q is the production of neutrons or photons by sources;
- $\nabla \equiv$ grad, operates with respect to r only.

The scattering function is so normalized that[1]

$$\int f(r \cdot \Omega')d_2\Omega' = 1. \tag{1.2}$$

We investigate the decomposition of Eq. (1.1) under the following simplifying assumptions:

$$\begin{aligned} &\text{a) } f = 1/(4\pi)\text{; i.e., we have isotropic scattering;}\\ &\text{b) } \sigma \text{ is constant;}\\ &\text{c) } Q(r,\Omega,v) = 0. \end{aligned} \tag{1.3}$$

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[1] Subsequently, the operator $\int \cdot d_2\Omega'$ will mean integration over the entire unit sphere in the sense defined in Sections 3 and 4.

Let the mean free path $1/\sigma$ be introduced as unit length. This is equivalent to setting $\sigma = 1$ in Eq. (1.1). Since $v \neq 0$, Eq. (1.1) reduces to

$$\Omega \cdot \nabla\psi + \psi - \frac{c}{4\pi} \int_{\{\Omega'\}} \psi(r, \Omega') d_2\Omega' = 0. \tag{1.4}$$

2. SEPARATION OF VARIABLES AND THE CHARACTERISTIC EQUATION

We look for all solutions that have the form

$$\psi_\Lambda(\mathbf{r},\Omega) = R_\Lambda(\mathbf{r})\phi_\Lambda(\Omega). \tag{2.1}$$

Substitution of Eq. (2.1) into Eq. (1.4) yields

$$-\frac{\Omega\cdot\nabla R_\Lambda}{R_\Lambda} = 1 - \frac{c}{4\pi\phi_\Lambda}\int\phi_\Lambda(\Omega')d_2\Omega' \tag{2.2}$$

under the assumption $R_\Lambda \neq 0$, $\phi_\Lambda \neq 0$. Since $\nabla R_\Lambda/R_\Lambda$ does not depend on r, it must be proportional to a parameter vector $-\Lambda$, which will be in general a complex vector,

$$\nabla R_\Lambda/R_\Lambda = -\Lambda, \tag{2.3}$$

and therefore

$$R_\Lambda = \text{const}\cdot e^{-(\Lambda\cdot \mathbf{r})}. \tag{2.4}$$

Therefore, the right-hand side of Eq. (2.2) is equal to $\Omega\cdot\Lambda$, and Eq. (2.2) takes the form

$$(1 - \Omega\cdot\Lambda)\phi_\Lambda - \frac{c}{4\pi}\int\phi_\Lambda(\Omega')d_2\Omega' = 0. \tag{2.5}$$

We normalize ϕ_Λ such that

$$\int \phi_\Lambda(\Omega')d\Omega' = \kappa, \tag{2.6}$$

where κ is a conveniently chosen, but fixed normalization constant (e.g., $\kappa = 1$, or $4\pi/c$). Then, from Eq. (2.5), if the solution is denoted by ϕ_Λ^0,

$$\phi_\Lambda^0 = \frac{\kappa c}{4\pi}\,\frac{1}{1 - \Omega\cdot\Lambda} \tag{2.7}$$

under the conditions that

$$\Omega\cdot\Lambda \neq 1 \quad \text{for all } \Omega \tag{2.8}$$

and Eq. (2.6) be satisfied. Substitution of Eq. (2.7) in Eq. (2.6) yields a condition for Λ, namely, the <u>characteristic equation,</u>

$$1 - \frac{c}{4\pi}\int\frac{d_2\Omega}{1 - \Omega\cdot\Lambda} = 0. \tag{2.9}$$

Definitions 2.1

The set of solutions of Eq. (2.9) satisfying Eq. (2.8) is denoted by $\{\Lambda^0\}$, and the set of corresponding functions ϕ^0_Λ will be called regular eigenfunctions of Eq. (2.5).

Hence, we can write for Eq. (2.1)

$$\psi^0_\Lambda = e^{-\Lambda \cdot r} \phi^0_\Lambda(\Omega), \quad \Lambda \in \{\Lambda^0\} \tag{2.10}$$

and ϕ^0_Λ is given by Eq. (2.7). The function $\psi^0_\Lambda(r,\Omega)$ has been so normalized that $\psi^0_\Lambda(0,\Omega) = \phi^0_\Lambda(\Omega)$.

Consider now the case where, for a given Λ, Eq. (2.8) is not satisfied. Then, at a certain direction $\Omega = \Omega^0$, ϕ_Λ in Eq. (2.7) is unbounded, and the meaning of the integral (2.6) must be redefined. Assuming this is done, we then obtain another characteristic equation with possible solutions for Λ, and a set of elementary functions corresponding to Eq. (2.10), but unbounded in Ω. These functions are, however, a subset of an even wider class of elementary solutions of the general form (2.1), when we extend the class of admissible functions to be generalized functions. We extend our function space by the set of functions $\delta(\Omega - \Omega_0)$ having the following definition.

Definition 2.2

Given any function $G(\Omega)$ which is continuous with respect to $\Omega, |\Omega| = 1$, in a neighborhood $D(\Omega)$ of a given direction Ω_0, then $\delta(\Omega - \Omega_0)$ is defined by

$$\int_{D(\Omega)} G(\Omega)\delta(\Omega - \Omega_0) d_2\Omega = \begin{cases} G(\Omega_0) & \text{for } \Omega_0 \in D(\Omega) \\ 0 & \text{for } \Omega_0 \notin D(\Omega) \end{cases}.$$

Now, we have the improper eigenfunctions (generalized eigenfunctions) as

$$\phi_\Lambda = \frac{\kappa c}{4\pi} \frac{1}{1 - \Omega \cdot \Lambda} + \kappa G \delta(\Omega - \Omega_0), \tag{2.11}$$

where

$$\Lambda \in \{\Lambda : 1 - \Omega_0 \cdot \Lambda = 0\}, \tag{2.12}$$

and G is a function possibly of Λ and Ω. Substitution of Eq. (2.11) into Eq. (2.5) under consideration of Eq. (2.6), and integration with respect to Ω over the unit sphere, show that the ϕ's of Eq. (2.11) are weak solutions indeed. The function G is defined by substituting Eq. (2.11) into Eq. (2.6). Hence,

$$G = 1 - \frac{c}{4\pi} \int \frac{d_2\Omega}{1 - \Omega \cdot \Lambda}. \tag{2.13}$$

An elementary solution corresponding to Eq. (2.11) is

$$\psi_\Lambda = e^{-\Lambda \cdot r} \phi_\Lambda, \tag{2.14}$$

where ϕ_Λ is given by Eq. (2.11), and Λ is an element from the set (2.12). The function ψ_Λ is so normalized that $\psi_\Lambda(0, \Omega) = \phi_\Lambda(\Omega)$.

Any linear combination of solutions of the forms of Eqs. (2.10) and (2.14) are again solutions of the linear Eq. (1.4). Therefore, the most general solution of Eq. (1.4) which can be represented by a linear combination of the regular eigenfunctions $\psi_\Lambda^0(\Omega)$ of Eq. (2.10) and the improper eigenfunctions $\psi_\Lambda(\Omega)$ of Eq (2.14) is given by

$$\psi(r, \Omega) = \int_{\{\Lambda^0\}} A^0(\Lambda)\, \psi_\Lambda^0\, d_4\Lambda + \int_{\{\Omega^0\}} d_2\Omega_0 \int_{\{\Lambda : \Omega_0 \Lambda = 1\}} d_4\Lambda A(\Lambda) \psi_\Lambda, \tag{2.15}$$

where $\{\Lambda^0\}$ is given by Eq. (2.9), and $\{\Omega^0\}$ is the set of all real directions. The first and second terms on the right-hand side of Eq. (2.15) are given explicitly by Eqs. (5.12) and (6.9), respectively, below.

Conjecture

Every solution of the homogeneous Eq. (1.4) in the class of functions which are integrable[2] with respect to Ω, and continuous in $|r| < \infty$ have a representation of the form of Eq. (2.15).

The following sections will be specific as to the meaning of integration with respect to Ω and Λ.

---

[2] See footnote 1.

3. INTEGRATION OF $(1 - \Omega \cdot \Lambda)^{-1}$

A. Geometrical Considerations

To evaluate Eqs. (2.9) and (2.13), a specific meaning is required for

$$I_\Lambda = \int \frac{d_2 \Omega}{1 - \Omega \Lambda} \tag{3.1}$$

This integral is invariant under the rotation group $\mathfrak{R}$ of Ω. The complex vector Λ can be represented as the sum of a real vector $\Lambda_1 \in E_3$ and an imaginary vector $i\Lambda_2$ with $\Lambda_2 \in E_3$; that is,

$$\Lambda = \Lambda_1 + i\Lambda_2. \qquad \text{(Definition)} \tag{3.2}$$

The integral I_Λ is therefore only a function of

$$\left.\begin{aligned} \lambda_1 &= |\Lambda_1|; \\ \lambda_2 &= |\Lambda_2|; \qquad\qquad \text{(Definition)} \\ \gamma &= \cos^{-1}[(\Lambda_1 \cdot \Lambda_2)/(\Lambda_1 \cdot \Lambda_2)] \end{aligned}\right\} \tag{3.3}$$

It turns out to be convenient to choose Cartesian coordinates such that

$$\left.\begin{aligned} \Lambda_1 &= \lambda_1 \{0, 0, 1\}; \\ \Lambda_2 &= \lambda_2 \{\sin\gamma, 0, \cos\gamma\}; \\ \Omega &= \{\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta\}. \end{aligned}\right\} \tag{3.4}$$

This means that Λ_1 lies on the z-axis, Λ_2 lies in the x,z-plane, and Ω is a point on the unit sphere, with θ and ϕ representing spherical coordinates (see Figure 3.1).

We define[3] the inner product $\Omega \cdot \Lambda \equiv (\Omega, \Lambda)$ to be

$$\left.\begin{aligned} \Omega \cdot \Lambda &= \Omega \cdot \Lambda_1 + i\Omega \cdot \Lambda_2; \\ \Omega \cdot \Lambda_1 &= \lambda_1 \cos\theta; \\ \Omega \cdot \Lambda_2 &= \lambda_2 (\cos\phi \sin\theta \sin\gamma + \cos\theta \cos\gamma). \end{aligned}\right\} \tag{3.5}$$

The integral I_Λ, expressed by Eq. (3.1), becomes improper when

$$1 - \Omega \cdot \Lambda = 0. \tag{3.6}$$

Separating real and imaginary parts, we obtain the following lemma.

---

3 Note that this definition differs from the usual definition in complex inner product spaces, where $\Omega \cdot \Lambda = \Omega \cdot \Lambda_1 - i\Omega \cdot \Lambda_2$.

Lemma 3.1

Given: $\Lambda = \Lambda_1 + i\Lambda_2$. Then the integral I_Λ, expressed by Eq. (3.1), is improper if and only if

$$\begin{aligned} \Omega \cdot \Lambda_1 &= 1, \\ \Omega \cdot \Lambda_2 &= 0, \end{aligned} \tag{3.7}$$

for some $\Omega \in \{\Omega : |\Omega| = 1\}$.

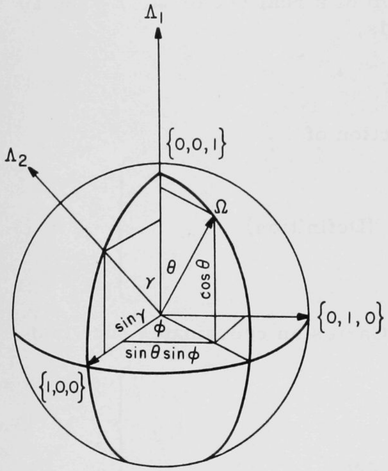

Fig. 3.1

Reference System for the Computation of $\int \frac{d\Omega}{1 - \Omega \cdot \Lambda}$

Corollary 3.1

A sufficient condition for I_Λ to be proper is $|\Lambda_1| < 1$. This corollary is evident when we write, by Eqs. (3.5),

$$\Omega \cdot \Lambda_1 \equiv \lambda_1 \cos\theta = 1,$$

which has no solution for $|\Lambda_1| < 1$.

We denote the directions Ω which satisfy Eqs. (3.7) by Ω_0, with the corresponding polar coordinates θ_0 and ϕ_0. Then, by Eqs. (3.5),

$$\lambda_1 \cos\theta_0 = 1; \tag{3.8}$$

$$\lambda_2(\cos\phi_0 \sin\theta_0 \sin\gamma + \cos\theta_0 \cos\gamma) = 0. \tag{3.9}$$

Equation (3.8) can be satisfied for any $\lambda_1 \geq 1$.

Equation (3.9) is satisfied when

i) $\lambda_2 = 0;$ (3.10)

and/or

ii) $-\cos \phi_0 = (\cos \theta_0/\sin \theta_0)(\cos \gamma/\sin \gamma).$ (3.11)

If Eq. (3.11) has a solution ϕ_0, then $-\phi_0$ is also a solution of Eq. (3.11).

We derive now the conditions under which Eqs. (3.11) and (3.8) are simultaneously satisfied. We have, from Eq. (3.8),

$$\cos \theta_0 = 1/\lambda_1 > 0. \tag{3.12}$$

Because $0 \leq \gamma \leq \pi$,

$$\sin \gamma \geq 0 \tag{3.13}$$

and because of Eqs. (3.11) and (3.12),

$$\cos^2 \phi_0 = [1/(\lambda_1^2 - 1)](\cos^2 \gamma/\sin^2 \gamma) \leq 1. \tag{3.14}$$

Therefore, from the second and third parts of Eq. (3.14),

$$\cos^2 \gamma + \sin^2 \gamma \leq \lambda_1^2 \sin^2 \gamma.$$

Finally, because of Eq. (3.13),

$$\lambda_1 \sin \gamma \geq 1. \tag{3.15}$$

We have the following corollary.

Corollary 3.2

Given: $\Lambda = \Lambda_1 + i\Lambda_2$. Then the integral I_Λ, expressed by Eq. (3.1), is improper if and only if

$\lambda_1 \geq 1;$ (where $\lambda_1 = |\Lambda_1|$)

and

i) $\lambda_2 = 0,$ (where $\lambda_2 = |\Lambda_2|$) (3.16)

and/or

(ii) $\lambda_1 \sin \gamma \geq 1.$ $\left(\text{where } \cos \gamma = \dfrac{\Lambda_1 \cdot \Lambda_2}{|\Lambda_1||\Lambda_2|}\right)$ (3.17)

These results can be obtained by purely geometrical considerations. The condition $\Omega \cdot \Lambda_1 = 1$ is represented by the intersection of the unit sphere $|\Omega| = 1$ and a plane perpendicular to Λ_1 with distance $1/\lambda_1$ from the origin. The condition $\Omega \cdot \Lambda_2 = 0$ represents the intersection of the unit sphere and the plane through the origin and perpendicular to Λ_2 (see Figure 3.2). It is evident that there are, for the system of Eqs. (3.8) and (3.11),

$$\left.\begin{array}{l} \text{no solutions when } \sin\gamma < \cos\theta_0 \\ \text{one solution when } \sin\gamma = \cos\theta_0 \\ \text{two solutions when } \sin\gamma > \cos\theta_0 \end{array}\right\} = 1/\lambda_1. \tag{3.18}$$

From Figure (3.2), one can also deduce Eq. (3.11),

$$\cos(\pm\phi_0) = -\cot\theta_0 \cot\gamma. \tag{3.19}$$

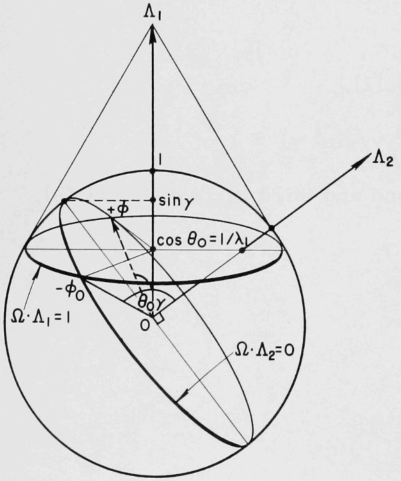

Fig. 3.2

Geometric Solution of
$1 - \Omega_0 \cdot \Lambda = 0$;
$\Omega_0 = (\theta_0, \pm\phi_0)$

We proceed now with the actual evaluation of I_Λ, expressed by Eq. (3.1). From Eq. (3.5), we conclude that

$$I_\Lambda = \int \frac{d_2\Omega}{1 - \Omega\cdot\Lambda} = \int_0^\pi \sin\theta \, d\theta \int_{-\pi}^{\pi} \frac{d\phi}{\alpha + \beta\cos\phi}, \tag{3.20}$$

where

$$\left.\begin{array}{l} \alpha = 1 - \lambda_1\cos\theta - i\lambda_2\cos\gamma\cos\theta; \\ \beta = \qquad\qquad\quad - i\lambda_2 \sin\gamma\sin\theta. \end{array}\right\} \tag{3.21}$$

First we evaluate

$$I_\theta = \int_{-\pi}^{+\pi} \frac{d\phi}{\alpha + \beta \cos\phi}; \tag{3.22}$$

then we evaluate

$$I_\Lambda = \int_0^\pi I_\theta \sin\theta \, d\theta. \tag{3.23}$$

B. Evaluation of I_θ

We introduce the transformation

$$u = \tan \phi/2, \qquad d\phi = 2du/(1+u^2). \tag{3.24}$$

Then,

$$I_\theta = \int_{-\pi}^{+\pi} \frac{d\phi}{\alpha + \beta \cos\phi} = \int_{-\infty}^{+\infty} \frac{1}{\alpha + \beta \dfrac{1-u^2}{1+u^2}} \cdot \frac{2\,du}{1+u^2} \tag{3.22}$$

$$= 2\int_{-\infty}^{+\infty} \frac{du}{(\alpha+\beta) + (\alpha-\beta)u^2} \tag{3.25}$$

$$= \frac{2}{\alpha-\beta} \int_{-\infty}^{+\infty} \frac{du}{u^2 + \dfrac{\alpha+\beta}{\alpha-\beta}}, \qquad \text{if } \alpha \neq \beta. \tag{3.26}$$

The zeros of the denominator under the integral in Eq. (3.26) are either a) complex numbers,

$$u^+ = i \sqrt[+]{\frac{\alpha+\beta}{\alpha-\beta}}, \qquad u^- = -u^+, \tag{3.27}$$

or b) real numbers, $\pm u^0$.

We require the "positive" square root of a complex number z^2 to have a positive real part, such that

$$\sqrt[+]{z^2} \Rightarrow \mathrm{Re}\, z > 0. \tag{3.28}$$

If Re z = 0, the square root is called "imaginary"; if Re z < 0, the square root is called "negative." We consider first the case where u^+ is complex, then the case when the zeros of the denominator in Eq. (3.26) are real.

a) <u>The integrand of I_θ has complex singularities.</u>

In this case, there is exactly one zero of the denominator, viz., u^+, in the upper half of the complex plane. We can use the residue theorem to determine I_θ from Eq. (3.26). That is,

$$I_\theta = \frac{2}{\alpha - \beta} \cdot \frac{2\pi i}{2u^+} = \frac{1}{\alpha - \beta} \cdot \frac{2\pi i}{i\sqrt[+]{\dfrac{\alpha + \beta}{\alpha - \beta}}}.$$

If we define

$$\sqrt{\alpha^2 - \beta^2} \equiv (\alpha - \beta) \sqrt[+]{\frac{\alpha + \beta}{\alpha - \beta}}, \tag{3.29}$$

we have

$$I_\theta = \int_{-\pi}^{+\pi} \frac{d\phi}{\alpha + \beta \cos\phi} = \frac{2\pi}{\sqrt{\alpha^2 - \beta^2}}. \tag{3.30}$$

We consider the case when $\beta = 0$. Then, from Eq. (3.22),

$$I_\theta = 2\pi/\alpha. \tag{3.31}$$

We see that this case is included in Eq. (3.30), since $\sqrt{\alpha^2 - 0} \equiv \alpha \cdot \sqrt[+]{1}$ according to Eq. (3.29), and so we may assume below that $\beta \neq 0$, or equivalently

$$\lambda_2 \sin\gamma \sin\theta \neq 0. \tag{3.32}$$

b) <u>The integrand of I_θ has real singularities.</u>

We distinguish three cases:

(i) $u^0 \neq 0$ and bounded.

(ii) $u^0 = 0$.

(iii) u^0 is unbounded.

(i) $u^0 \neq 0$ and bounded.

In this case, $(u^0)^2 = (-u^0)^2 > 0$.

Since

$$u^0 = \pm i \sqrt{\frac{\alpha + \beta}{\alpha - \beta}},$$

the inequality $(u^0)^2 > 0$ is equivalent to

$$(\alpha + \beta)/(\alpha - \beta) < 0. \tag{3.33}$$

To derive the conditions under which this is true, we write

$$\left.\begin{aligned} \alpha + \beta &= (1 - \lambda_1 \cos\theta) - i\lambda_2 (\cos\gamma \cos\theta + \sin\gamma \sin\theta); \\ \alpha - \beta &= (1 - \lambda_1 \cos\theta) - i\lambda_2 (\cos\gamma \cos\theta - \sin\gamma \sin\theta). \end{aligned}\right\} \tag{3.34}$$

Then

$$\begin{aligned} (\alpha + \beta)/(\alpha - \beta) &= (\alpha + \beta)(\overline{\alpha - \beta})/(|\alpha - \beta|^2) \\ &= \frac{(1 - \lambda_1 \cos\theta)^2 + \lambda_2^2 (\cos^2\gamma \cos^2\theta - \sin^2\gamma \sin^2\theta) - 2i\lambda_2 (1 - \lambda_1 \cos\theta) \sin\gamma \sin\theta}{(1 - \lambda_1 \cos\theta)^2 + \lambda_2^2 \cos^2(\gamma + \theta)}. \end{aligned} \tag{3.35}$$

For this expression to be real, we have, under consideration of (3.32), the necessary condition

$$1 - \lambda_1 \cos\theta = 0. \tag{3.36}$$

For (3.35) to be negative, it is necessary that

$$\cos^2\gamma \cos^2\theta - \sin^2\gamma \sin^2\theta < 0,$$

or equivalently

$$\sin^2\gamma > \cos^2\theta. \tag{3.37}$$

If we denote by θ_0 the angle for which Eq. (3.36) is satisfied, and consider that $0 < \gamma < \pi$, Eq. (3.37) yields

$$\sin\gamma > \cos\theta_0, \tag{3.38}$$

or

$$\lambda_1 \sin\gamma > 1, \tag{3.39}$$

which corresponds to the third condition of (3.18), as the reader may have anticipated.

Assume now that (3.36) and (3.39) are satisfied; then it is seen that $u^0 \neq 0$ and bounded, and from Eq. (3.26) we obtain

$$I_\theta = \frac{2}{\alpha - \beta}\int_{-\infty}^{+\infty}\frac{du}{(u-u^0)(u+u^0)}$$

$$= \frac{2}{\alpha-\beta}\cdot\frac{1}{2u^0}\left\{\int_{-\infty}^{+\infty}\frac{du}{u-u^0} - \int_{-\infty}^{+\infty}\frac{du}{u+u^0}\right\} = 0, \tag{3.40}$$

where we applied Cauchy's Principal Value Theorem to the integrals, and each integral is zero by itself.

(ii) $\underline{u^0 = 0}$

In this case, we argue that

$$(\alpha+\beta)/(\alpha-\beta) = 0. \tag{3.41}$$

We can proceed as in Section (i) above and obtain the following necessary conditions for u^0 to be zero:

$$1 - \lambda_1\cos\theta = 0; \tag{3.42}$$

$$\lambda_1 \sin\gamma = 1. \tag{3.43}$$

However, the conditions $\alpha + \beta = 0$, $\alpha - \beta \neq 0$, yield the additional condition

$$\cos\gamma = -\sin\theta < 0. \tag{3.44}$$

Conversely, from conditions (3.42) and (3.44) follows $u^0 = 0$; and from Eq. (3.26), we obtain

$$I_\theta = \frac{2}{\alpha-\beta}\int_{-\infty}^{+\infty}\frac{du}{u^2} \to \infty; \tag{3.45}$$

i.e., the integral does not exist.

If we denote the angle which satisfies (3.42) by θ_0, then conditions (3.42) and (3.43) correspond to the second condition of Eq. (3.18); and from (3.44) and (3.19), we obtain the double value $\pm\phi^0 = 0$. Hence the unboundedness of I_θ is not surprising in this case.

(iii) u^0 is unbounded.

In this case, we argue that

$$(\alpha - \beta)/(\alpha + \beta) = 0, \tag{3.46}$$

and we proceed as in Section (ii) above to obtain the necessary and sufficient conditions

$$\left.\begin{aligned} 1 - \lambda_1 \cos\theta &= 0, \\ \lambda_1 \sin\gamma &= 1; \end{aligned}\right\} \tag{3.47}$$

$$\cos\gamma = \sin\theta. \tag{3.48}$$

Then, from Eq. (3.25),

$$I_\theta = \frac{2}{\alpha + \beta} \int_{-\infty}^{+\infty} \frac{du}{1 + 0 \cdot u^2} \to \infty; \tag{3.49}$$

i.e., the integral does not exist.

Again, we can find the geometrical equivalent to the conditions (3.47) in the second equation of (3.18); and Eq. (3.19) yields a double value $\phi^0 = \pm\pi$. Eqs. (3.47) are the same as Eqs. (3.42) and (3.43).

Since it follows from condition (3.47) that

$$(\alpha + \beta)(\alpha - \beta) = 0,$$

Eqs. (3.45) and (3.49) can be considered limit cases of Eq. (3.30). Therefore we can summarize the results of Section B in the following equation:

$$I_\theta = \int_{-\pi}^{+\pi} \frac{d\phi}{\alpha + \beta\cos\phi} = \begin{cases} \dfrac{2\pi}{\sqrt{\alpha^2 - \beta^2}} & \text{for } \begin{cases} \lambda_1 \cos\theta \neq 1 \\ \lambda_1 \cos\theta = 1 \text{ and } \lambda_1 \sin\gamma \leq 1 \end{cases} \\ 0 & \text{for } \quad \lambda_1 \cos\theta = 1 \text{ and } \lambda_1 \sin\gamma > 1, \end{cases} \tag{3.50}$$

where α and β are defined by (3.21), and $\sqrt{\alpha^2 - \beta^2}$ is defined by (3.29), unless $\alpha^2 - \beta^2 = 0$.

4. FURTHER INTEGRATION OF $(1 - \Omega\cdot\Lambda)^{-1}$

A. The Function $\sqrt{Z(\mu)}$

In Section 3 we gave a closed expression for I_θ. We must now consider I_θ as a function of θ, and determine I_Λ as indicated in Eq. (3.23). It will be convenient below to consider I_θ as a function of $\cos\theta$. Therefore we introduce the notation

$$\left.\begin{aligned} \mu &\equiv \cos\theta; \\ \mu &\equiv \cos\theta_0 \equiv 1/\lambda_1, \qquad \text{for } \lambda_1 \geq 1; \\ I_\mu(\mu) &\equiv I_\theta(\theta). \end{aligned}\right\} \tag{4.1}$$

To set the proper background for the integration of I_μ, we discuss first the function

$$\sqrt{Z(\mu)} \equiv (\alpha - \beta) \sqrt[+]{\frac{\alpha + \beta}{\alpha - \beta}}. \tag{4.2}$$

From Eqs. (3.35) and (4.1), we have

$$\frac{\alpha + \beta}{\alpha - \beta} = \frac{(1 - \lambda_1\mu)^2 + \lambda_2^2(\mu^2 - \sin^2\gamma) - 2i\lambda_2(1 - \lambda_1\mu)\sin\gamma\sin\theta}{(1 - \lambda_1\mu)^2 + \lambda_2^2\cos^2(\gamma + \theta)} \tag{4.3}$$

as a function of μ. The definition of its square root was given in (3.28) such that the z^2-plane was slit along the negative axis. The behavior of the positive square root of Eq. (4.3) as a function of μ is as follows.

a) If $\underline{\lambda_1 < 1}$, it follows immediately from Eq. (4.3) that

$$\operatorname{Im}\,(\alpha + \beta)/(\alpha - \beta) < 0, \tag{4.4}$$

and hence $\sqrt[+]{(\alpha + \beta)/(\alpha - \beta)}$ is continuous in the interval $-1 \leq \mu \leq +1$.

b) If $\underline{\lambda_1 \geq 1}$, it follows from Eq. (4.3) that, respectively,

$$\operatorname{Im}\frac{\alpha + \beta}{\alpha - \beta}\begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 0 \qquad \text{for } \mu \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} \mu_0. \tag{4.5}$$

At $\mu = \mu_0$,

$$\operatorname{Re}\frac{\alpha + \beta}{\alpha - \beta}\begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 \qquad \text{for } \lambda_1 \sin\gamma \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 1. \tag{4.6}$$

We set

$$\rho = \left| \sqrt{\frac{\mu_0^2 - \sin^2\gamma}{\cos^2(\gamma+\theta)}} \right|. \tag{4.7}$$

As μ approaches μ_0 from both sides, we have in the limit

$$+\sqrt{\frac{\alpha+\beta}{\alpha-\beta}} = \begin{cases} \rho & \text{for } \mu = \mu_0 \pm 0 \text{ and } \lambda_1 \sin\gamma < 1; \\ i\rho & \text{for } \mu = \mu_0 - 0 \\ -i\rho & \text{for } \mu = \mu_0 + 0 \end{cases} \left.\vphantom{\begin{matrix}a\\b\end{matrix}}\right\} \text{ and } \lambda_1 \sin\gamma > 1. \tag{4.8}$$

The square root is not defined for $\lambda_1 \sin\gamma = 1$. Hence, (4.8) is continuous in the interval $-1 \le \mu \le 1$ for $\lambda_1 \sin\gamma < 1$, but has a discontinuity at $\mu = \mu_0$ in the case $\lambda_1 \sin\gamma > 1$.

Since $(\alpha - \beta)$ is a continuous function in μ, we can conclude that $\sqrt{Z(\mu)}$ is continuous in $-1 \le \mu \le 1$ for $\{\lambda_1 < 1\}$, and $\{\lambda_1 \ge 1,\ \lambda_1 \sin\gamma < 1\}$; but has a discontinuity at $\mu = \mu_0$ for $\{\lambda_1 > 1,\ \lambda_1 \sin\gamma > 1\}$. In the case of $\lambda_1 \sin\gamma = 1$, $\sqrt{Z(\mu)} = 0$, and $\sqrt{Z(\mu)}$ is therefore also continuous.

We are now in a position to attribute signs to $\sqrt{Z(\mu)}$.

From Eqs. (4.2) and (3.29), it follows that

$$\sqrt{Z(\mu)} \equiv \sqrt{\alpha^2 - \beta^2} \equiv \sqrt{(\alpha+\beta)(\alpha-\beta)}, \tag{4.9}$$

with the sign of the square root determined by Eq. (4.2). The function $\sqrt{Z(\mu)}$ can only change its sign at a point μ where $\operatorname{Re}\sqrt{Z(\mu)} = 0$, or where $\sqrt{Z(\mu)}$ is discontinuous. The first alternative is equivalent to the condition that both

$$\left.\begin{aligned} X &= \operatorname{Re} Z(\mu) < 0 \\ &\text{and} \\ Y &= \operatorname{Im} Z(\mu) = 0, \end{aligned}\right\} \tag{4.10}$$

or that Z crosses the negative axis.

From Eqs. (3.34) and (4.1), it follows for $Z = X + iY$ that

$$\left.\begin{aligned} X &= (1 - \lambda_1\mu)^2 + \lambda_2^2(\sin^2\gamma - \mu^2); \\ Y &= -2\lambda_2 \cos\gamma\,\mu(1 - \lambda_1\mu). \end{aligned}\right\} \tag{4.11}$$

We enumerate now the cases where $Y = 0$.

(i) $\underline{\mu = 0.}$ If $\mu = 0$, then $Y = 0$ but $X > 0$. Hence Eq. (4.10) is not satisfied, and $\sqrt{Z}$ does not change sign at $\mu = 0$.

(ii) $\underline{1 - \lambda_1\mu = 0.}$ This case is only possible when $\lambda_1 \geq 1$. Then $\mu = \mu_0$, and $X < 0$ requires $\lambda_1 \sin\gamma < 1$. Hence $\sqrt{Z}$ does change its sign for $\lambda_1 \geq 1$, and $\lambda_1 \sin\gamma < 1$. When $\lambda_1 \geq 1$, and $\lambda_1 \sin\gamma > 1$, $\sqrt{Z}$ has a discontinuity at $\mu = \mu_0$ and will change its sign also.

(iii) $\underline{\lambda_2 \cos\gamma = 0.}$ In this case, Y is identically zero. If $\lambda_2 = 0$, then

$$\sqrt{Z} = 1 - \lambda_1\mu \tag{4.12}$$

and therefore changes its sign only when $\lambda_1 > 1$ at $\mu = \mu_0$.

If $\cos\gamma = 0$, then $X = (1 - \lambda_1\mu)^2 + \lambda_2^2(1 - \mu^2) > 0$, and $\sqrt{Z}$ does not change its sign.

(iv) We can also conclude from (4.11) that $Z = 0$ for $-1 \leq \mu \leq 1$ only if $\mu = \mu_0$ and $\lambda_1 \sin\gamma = 1$, as already noted in Section 3.

The sign of the square root of $Z(\mu)$ is now determined by substituting $\mu = \pm 1$ in Eq. (4.2). We obtain immediately from Eq. (3.34) that

$$(\alpha + \beta)/(\alpha - \beta) = 1, \qquad \text{for } \mu = \pm 1; \tag{4.13}$$

and

$$\alpha - \beta = (1 \mp \lambda_1) \mp i\lambda_2 \cos\gamma, \qquad \text{for } \mu = \pm 1. \tag{4.14}$$

Hence, by the discussion above, we have for

$$\underline{\lambda_1 < 1,} \qquad \sqrt{Z} = \sqrt[+]{Z} \qquad \text{for } -1 \leq \mu \leq 1; \tag{4.15}$$

and for $\underline{\lambda_1 \geq 1,}$

$$\sqrt{Z} = \begin{cases} \sqrt[+]{Z} & \text{for } -1 \leq \mu < \mu_0 \\ \sqrt[-]{Z} & \text{for } \mu_0 < \mu \leq 1, \end{cases} \tag{4.16}$$

$$\sqrt[+]{Z(\mu_0 - 0)} = \sqrt[-]{Z(\mu_0 + 0)} \qquad \text{for } \lambda_1 \sin\gamma < 1, \tag{4.16a}$$

$$\sqrt[+]{Z(\mu_0 - 0)} = -\sqrt[-]{Z(\mu_0 + 0)} \qquad \text{for } \lambda_1 \sin\gamma > 1, \tag{4.16b}$$

$$\sqrt{Z(\mu_0 \pm 0)} = 0 \qquad \text{for } \lambda_1 \sin\gamma = 1; \tag{4.16c}$$

where the last three equations are a consequence of Eq. (4.8) and Remark (iv) above.

We are now ready to integrate I_μ with respect to μ.

B. Integration of I_μ

Comparison of Eqs. (3.50), (4.1), and (4.9) shows that

$$I_\mu = 2\pi/\sqrt{Z(\mu)} \tag{4.17}$$

for $\lambda_1\mu \neq 1$, $\lambda_1\mu = 1$ and $\lambda_1 \sin\gamma < 1$.

For $\lambda_1 \sin\gamma = 1$ and $\lambda_1\mu \to 1$, I_μ is unbounded. For $\lambda_1 \sin\gamma > 1$ and $\lambda_1\mu = 1$, I_μ is zero. If we use Riemannian integration, we have, under consideration of Eqs. (4.15), (4.16), and (3.50), two cases, a) and b), which can be integrated.

a) $\lambda_1 < 1$, or

$\lambda_1 \geq 1$ and $\lambda_1 \sin\gamma < 1$:

$$I_\Lambda = 2\pi \int_{-1}^{+1} \frac{d\mu}{\sqrt{Z(\mu)}} . \tag{4.18}$$

b) $\lambda_1 \geq 1$, $\lambda_1 \sin\gamma > 1$:

$$I_\Lambda = 2\pi \left\{ \int_{-1}^{\mu_0 - 0} \frac{d\mu}{+\sqrt{Z(\mu)}} + \int_{\mu_0 + 0}^{1} \frac{d\mu}{-\sqrt{Z(\mu)}} . \right\} \tag{4.19}$$

The situation $\lambda_1 \geq 1$, $\lambda_1 \sin\gamma = 1$, leads to an improper integral which must be obtained by a limiting process. This process is carried through in Section C.d) below.

To obtain an analytic expression for the desired integral, we observe from Eq. (4.11) that $Z = X + iY$ can be written as

$$Z(\mu) = a\mu^2 - 2b\mu + c, \tag{4.20}$$

where

$$\left. \begin{aligned} a &= \lambda_1^2 - \lambda_2^2 + 2i\lambda_1\lambda_2 \cos\gamma, \\ b &= \lambda_1 + i\lambda_2 \cos\gamma, \\ c &= 1 + \lambda_2^2 \sin^2\gamma. \end{aligned} \right\} \tag{4.21}$$

Then the indefinite integral of $Z(\mu)^{-1/2}$ is

$$\int \frac{d\mu}{\sqrt{Z}} = \frac{1}{\sqrt{a}} \log[a\mu - b + \sqrt{a}\ \sqrt{Z}\,] + \text{const}, \tag{4.22}$$

which can be easily verified.

The sign of $\sqrt{a}$ will be so chosen that the right-hand side of Eq. (4.22) is bounded when $Z^{-1/2}$ is bounded.

a) To obtain the integral (4.18), we must evaluate $\sqrt{Z}$ at $\mu = \pm 1$ under the conditions (4.15) and (4.16). This can be done using Eqs. (4.20) and (4.21), or more simply by noting Eqs. (3.34). Then

$$\begin{aligned} Z(\pm 1) &= [(\alpha + \beta)(\alpha - \beta)]_{\mu} = \pm 1 \\ &= (1 \mp \lambda_1 \mp i\lambda_2 \cos\gamma)^2 \\ &= (1 \mp b)^2. \end{aligned}$$

Hence, by Eqs. (4.15) and (4.16),

$$\sqrt{Z(\pm 1)} = 1 \mp b, \tag{4.23}$$

and by (4.22), we obtain for (4.18)

$$\begin{aligned} \int_{-1}^{+1} \frac{d\mu}{\sqrt{Z}} &= \frac{1}{\sqrt{a}} \log \frac{a - b + \sqrt{a}(1 - b)}{-a - b + \sqrt{a}(1 + b)} \\ &= \frac{1}{\sqrt{a}} \log \frac{(1 + \sqrt{a})(\sqrt{a} - b)}{(1 - \sqrt{a})(\sqrt{a} - b)}. \end{aligned} \tag{4.24}$$

We assign the negative sign to $\sqrt{a}$. In this case, even for $\lambda_2 \to 0$, $(\sqrt{a} - b) \neq 0$ and can be canceled.

Now we define the inner product $\Lambda^2 \equiv \Lambda \cdot \Lambda \equiv (\Lambda, \Lambda)$ to be, for

$$\left.\begin{aligned} \Lambda &= \Lambda_1 + i\Lambda_2, \\ \Lambda^2 &\equiv (\Lambda_1 + i\Lambda_2)(\Lambda_1 + i\Lambda_2) \\ &\equiv \Lambda_1^2 - \Lambda_2^2 + 2i\Lambda_1\Lambda_2 \\ &= \lambda_1^2 - \lambda_2^2 + 2i\lambda_1\lambda_2 \cos\gamma. \end{aligned}\right\} \tag{4.25}$$

Comparing this expression with (4.21), we see that

$$\Lambda^2 = a. \tag{4.26}$$

Therefore, substituting

$$\sqrt{a} = \bar{\sqrt{a}} = -\overset{+}{\sqrt{a}} \equiv -\sqrt{\Lambda^2} \tag{4.27}$$

in Eq. (4.24), yields[4]

$$I_\Lambda = \int \frac{d\Omega}{1 - \Omega\cdot\Lambda} = \int_{-1}^{+1} I_\mu d\mu = \frac{2\pi}{\sqrt{\Lambda^2}} \log \frac{1 + \sqrt{\Lambda^2}}{1 - \sqrt{\Lambda^2}} = \frac{4\pi}{\sqrt{\Lambda^2}} \tanh^{-1} \sqrt{\Lambda^2}$$

$$\text{for } \{\lambda_1 < 1\} \text{ and } \{\lambda_1 \geq 1,\ \lambda_1 \sin\gamma < 1\}. \tag{4.28}$$

where Λ^2 is defined by (4.25), and the Λ^2-plane is cut along the negative axis.

From the final equation, we see that the sign of $\sqrt{a}$ is of no practical interest; i.e., replacing $\sqrt{\Lambda^2}$ by $-\sqrt{\Lambda^2}$ leaves Eq. (4.28) unchanged.

b) To obtain the integral (4.19), we must evaluate $\sqrt{Z}$ at $\mu = \pm 1$ and $\mu = \mu_0 \pm 0$. We obtain, from Eqs. (4.11) and (4.16),

$$\sqrt{Z(\mu_0 \pm 0)} = \mp\lambda_2 \overset{+}{\sqrt{\sin^2\gamma - \mu_0^2}}, \tag{4.29}$$

while

$$\sqrt{Z(\pm 1)} = 1 \mp b, \tag{4.30}$$

as in Eq. (4.23). Therefore, by Eqs. (4.1), (4.21), and (4.26),

$$\int_{-1}^{\mu_0 - 0} \frac{d\mu}{\sqrt{Z}} + \int_{\mu_0 + 0}^{+1} \frac{d\mu}{\bar{\sqrt{Z}}} =$$

$$= \frac{1}{\sqrt{a}} \log \frac{a\mu_0 - b + \sqrt{a}\lambda_2 \sqrt{\sin^2\gamma - \mu_0^2}}{-a - b + \sqrt{a}(1 + b)} + \frac{1}{\sqrt{a}} \log \frac{a - b + \sqrt{a}(1 - b)}{a\mu_0 - b - \sqrt{a}\lambda_2\sqrt{\sin^2\gamma - \mu_0^2}}$$

$$= \frac{1}{\sqrt{a}} \log \frac{(1 + \sqrt{a})(\sqrt{a} - b)}{(1 - \sqrt{a})(\sqrt{a} - b)} \cdot \frac{\lambda_2[-\lambda_2/\lambda_1 + i\cos\gamma + (1/\lambda_1)\sqrt{a}\sqrt{\lambda_1^2 \sin^2\gamma - 1}]}{\lambda_2[-\lambda_2/\lambda_1 + i\cos\gamma - (1/\lambda_1)\sqrt{a}\sqrt{\lambda_1^2 \sin^2\gamma - 1}]} \tag{4.31}$$

$$= \frac{1}{\sqrt{a}} \log \frac{1 + \sqrt{\Lambda^2}}{1 - \sqrt{\Lambda^2}} \cdot \frac{-\lambda_2 + i\lambda_1 \cos\gamma + \sqrt{a}\sqrt{\lambda_1^2 \sin^2\gamma - 1}}{-\lambda_2 + i\lambda_1 \cos\gamma - \sqrt{a}\sqrt{\lambda_1^2 \sin^2\gamma - 1}} \tag{4.32}$$

We note that

$$-\lambda_2 + i\lambda_1 \cos\gamma = \bar{\sqrt{\lambda_1^2 \sin^2\gamma - a}}\,; \tag{4.33}$$

[4]Note that $\sqrt{\Lambda^2}$ is a complex scalar and not the vector Λ.

however, we do not use this identity in our final equation. Thus, from (4.32) and (4.19),

$$I_\Lambda = \int \frac{d_2\Omega}{1 - \Omega\cdot\Lambda} = \int_{-1}^{+1} I_\mu d\mu$$

$$= \frac{2\pi}{\sqrt{\Lambda^2}} \log \frac{\sqrt{\Lambda^2}+1}{\sqrt{\Lambda^2}-1} \cdot \frac{\sqrt{\Lambda^2}\sqrt{\lambda_1^2 \sin^2\gamma - 1} - \lambda_2 + i\lambda_1 \cos\gamma}{\sqrt{\Lambda^2}\sqrt{\lambda_1^2 \sin^2\gamma - 1} + \lambda_2 - i\lambda_1 \cos\gamma} \qquad (4.34)$$

for $\{\lambda_1 \geq 1, \text{ and } \lambda_1 \sin\gamma > 1\}$,

where Λ^2 is defined by (4.25), and the Λ^2-plane is cut along the negative axis.

Use of Eq. (4.33) would give Eq. (4.34) a more symmetric appearance, but no additional insight.

C. Limit Cases

The integral I_μ reduces for $\lambda_2 \sin\gamma = 0$ to Eq. (3.31).

a) If we set $\lambda_2 = 0$, then from (3.21)

$$I_\Lambda = 2\pi \int_{-1}^{+1} \frac{d\mu}{1 - \lambda_1} = \begin{cases} \dfrac{2\pi}{\lambda_1} \log \dfrac{1+\lambda_1}{1-\lambda_1} & \text{for } \lambda_1 \leq 1 \\[2ex] \dfrac{2\pi}{\lambda_1} \log \dfrac{\lambda_1+1}{\lambda_1-1} & \text{for } \lambda_1 \geq 1, \end{cases} \qquad (4.35)$$

according to results which were obtained in the case of plane geometry. The Cauchy Principle value was used to obtain the second result.

b) If we set $\sin\gamma = 0$, it follows from (3.21) that

$$I_\Lambda = 2\pi \int_{-1}^{+1} \frac{d\mu}{1 - (\lambda_1 \pm i\lambda_2)\mu} = \frac{2\pi}{\lambda_1 \pm i\lambda_2} \log \frac{1 + \lambda_1 \pm i\lambda_2}{1 - \lambda_1 \mp i\lambda_2}, \qquad (4.36)$$

where the upper sign corresponds to the value $\cos\gamma = +1$ and the lower sign corresponds to $\gamma = -1$.

If we let λ_2 go to zero in Eq. (4.36), we have

$$\lim_{\lambda_2 \to 0} I_\Lambda = \frac{2\pi}{\lambda_1} \log \left| \frac{1 + \lambda_1}{1 - \lambda_1} \right| + \begin{cases} 0 & \text{for } \lambda_1 < 1 \\ i\pi & \text{for } \cos \gamma = 1 \\ -i\pi & \text{for } \cos \gamma = -1 \end{cases} \left.\vphantom{\begin{matrix} a \\ b \end{matrix}}\right\} \text{and } \lambda_1 > 1. \tag{4.37}$$

Hence, I_Λ in Eq. (4.35) is the arithmetic mean of I_Λ in Eq. (4.37).

c) Now we investigate Eq. (4.28) and obtain, by (4.25),

$$\lim_{\lambda_2 \to 0} \frac{2\pi}{\sqrt{\Lambda^2}} \log \frac{1 + \sqrt{\Lambda^2}}{1 - \sqrt{\Lambda^2}} = \frac{2\pi}{\lambda_1} \log \left| \frac{1 + \lambda_1}{1 - \lambda_1} \right| + \begin{cases} 0 & \text{for } \lambda_1 < 1 \\ i\pi & \text{for } \cos \gamma > 0 \\ -i\pi & \text{for } \cos \gamma < 0 \end{cases} \left.\vphantom{\begin{matrix} a \\ b \end{matrix}}\right\} \text{and } \lambda_1 > 1. \tag{4.38}$$

Therefore, Eq. (4.37) is a special case of (4.38). The case $\sin \gamma = 0$ corresponds exactly to Eq. (4.37).

We consider now the case $\cos \gamma = 0$. This case will be important below. Note that $\cos \gamma = 0$ implies

$$\lambda_1 \sin \gamma = \lambda_1 < 1,$$

and therefore Eq. (4.28) reduces to

$$I_\Lambda = \frac{2\pi}{\sqrt{\lambda_1^2 - \lambda_2^2}} \log \frac{1 + \sqrt{\lambda_1^2 - \lambda_2^2}}{1 - \sqrt{\lambda_1^2 - \lambda_2^2}} \qquad \text{for } \cos \gamma = 0, \ \lambda_1 < 1. \tag{4.39}$$

As $\lambda_2 \to 0$, Eq. (4.39) reduces to Eqs. (4.37) and (4.35) for $\lambda_1 < 1$.

d) In a similar manner, we investigate Eq. (4.34), where $\lambda_1 \geq 1$ always. We have

$$\lim_{\lambda_2 \to 0} I_\Lambda = \frac{2\pi}{\lambda_1} \log \frac{\lambda_1 + 1}{\lambda_1 - 1} \cdot \frac{\sqrt{\lambda_1^2 \sin^2 \gamma - 1} + i \cos \gamma}{\sqrt{\lambda_1^2 \sin^2 \gamma - 1} - i \cos \gamma}. \tag{4.40}$$

The limit depends on the angle γ. For $\cos \gamma = 0$, Eq. (4.40) reduces to Eq. (4.35) for $\lambda_1 \geq 1$. One can also consider I_Λ in Eq. (4.35) as the mean value of I_Λ in Eq. (4.40) with respect to γ.

As $(\lambda_1 \sin \gamma - 1) \to 0+$,

$$I_\Lambda = \frac{2\pi}{\sqrt{\Lambda^2}} \log \frac{1 + \sqrt{\Lambda^2}}{1 - \sqrt{\Lambda^2}}. \tag{4.41}$$

Hence Eq. (4.34) changes continuously to Eq. (4.28). From Eqs. (4.40) and (4.41), we conclude

$$\lim_{\lambda_2\to 0}\ \lim_{\lambda_1 \sin\gamma\to 1+} I_\Lambda = \lim_{\lambda_1 \sin\gamma\to 1+}\ \lim_{\lambda_2\to 0} I_\Lambda, \tag{4.42}$$

and the result coincides with Eq. (4.38) for $\lambda_1 > 1$.

If $\gamma \to 0$, it follows from $\lambda_1 \sin\gamma > 1$ that $\lambda_1 \to \infty$. In this case, from Eq. (4.34), $\lim\limits_{\lambda_1\to\infty} I_\Lambda \to 0$.

Similarly

$$\lim_{\lambda_2\to\infty} I_\Lambda \to 0.$$

For the sake of completeness, we mention that the limit $\lambda_1 \to 0$ does not cause any difficulties, since in

$$\lim_{\lambda_1\to 0} \sqrt{\Lambda^2} = \pm i\lambda_2$$

either sign may be used in Eq. (4.28).

5. SOLUTION OF THE CHARACTERISTIC EQUATION AND THE REGULAR EIGENFUNCTIONS

In Eqs. (2.7) to (2.9), we defined the regular eigenfunctions and the characteristic equation. In Corollary 3.2, we asserted that eigenfunctions are regular only if

$$\left.\begin{array}{ll} \text{a)} & \lambda_1 < 1; \\ \text{or} & \\ \text{b)} & \lambda_1 \geq 1, \ \underline{\text{and}}\ \lambda_2 \neq 0, \ \underline{\text{and}}\ \lambda_1 \sin\gamma < 1. \end{array}\right\} \qquad (5.1)$$

We shall see below, in Eq. (5.6), that for regular eigenfunctions, $\lambda_1 < 1$ is always true.

In Section 4 we showed that the integral I_Λ, appearing in the characteristic Eq. (2.9) and satisfying conditions (5.1), is given by Eq. (4.28). Therefore, we can write the following expression for the characteristic equation.

<u>Characteristic Equation</u>

$$1 - (c/z)\tanh^{-1} z = 0; \qquad \sqrt{\Lambda^2} = z\,, \ \Lambda \in \{\Lambda^0\}. \qquad (5.2)$$

This is the same equation in z as was found for the characteristic equation in plane geometry, and has two solutions, λ_0 and $-\lambda_0$, for given c as follows:

$$\left.\begin{array}{lll} \text{a)} & 0 < c < 1 & \lambda_0 \text{ is real, and } \lambda_0^2 < 1; \\ \text{b)} & c = 1 & \lambda_0 = 0, \text{ and } \lambda_0^2 = 0; \\ \text{c)} & 1 < c < \infty & \lambda_0 \text{ is imaginary, and } \lambda_0^2 < 0. \end{array}\right\} \qquad (5.3)$$

Hence, λ_0 can be considered known for given c, and thus

$$\{\Lambda^0\} = \{\Lambda : \Lambda^2 = \lambda_0^2, \forall\Omega \rightarrow \Omega\Lambda \neq 1\}. \quad \text{(Definition)} \qquad (5.4)$$

This means that every vector Λ that satisfies $\Lambda^2 = \lambda_0^2$ and $\Omega\cdot\Lambda \neq 1$ for each Ω, is an element of $\{\Lambda^0\}$. Since λ_0^2 is real, we have, by (4.25), the condition

$$\lambda_1^2 - \lambda_2^2 = \lambda_0^2, \qquad \lambda_1\lambda_2\cos\gamma = 0, \qquad (5.5)$$

for $\Lambda^2 = \lambda_0^2$. The condition $\Omega\cdot\Lambda \neq 1$ for each Ω, yielded the conditions (5.1). These conditions impose only restrictions on the magnitudes λ_1 and λ_2 of the real and imaginary components Λ_1 and Λ_2, respectively, of $\Lambda = \Lambda_1 + i\Lambda_2$

and its mutual angle γ in E_3. From Eq. (5.5), we see that if $\lambda_1\lambda_2 \neq 0$, then $\cos\gamma = 0$. Therefore, by (5.1b),

$$\lambda_1 < 1. \tag{5.6}$$

We treat the three cases of (5.3) separately in the following paragraphs.

a) $\underline{0 < c < 1}$

From Eq. (5.5), we obtain

$$\lambda_1 = \sqrt{\lambda_2^2 + \lambda_0^2}; \tag{5.7a}$$

and from (5.3a) and (5.6), we obtain

$$\lambda_0 \leq \lambda_1 < 1; \qquad 0 \leq \lambda_2 < \sqrt{1 - \lambda_0^2}. \tag{5.7b}$$

b) $\underline{c = 1}$

By the same equations as used in a), we obtain

$$\lambda_1 = \lambda_2; \tag{5.8a}$$

$$\left.\begin{array}{l} 0 \leq \lambda_1 < 1; \\ 0 \leq \lambda_2 < 1. \end{array}\right. \tag{5.8b}$$

c) $\underline{1 < c < \infty}$

Again, by the same equations as used in a), we obtain

$$\lambda_1 = \sqrt{\lambda_2^2 - |\lambda_0|^2}; \tag{5.9a}$$

$$0 \leq \lambda_1 < 1; \qquad |\lambda_0| \leq \lambda_2 < \sqrt{1 + |\lambda_0|^2}. \tag{5.9b}$$

Because of (5.3c), we can write Eq. (5.9a) as

$$\lambda_1 = \sqrt{\lambda_2^2 + \lambda_0^2} \tag{5.9c}$$

to conform with Eq. (5.7a).

The relationship between λ_1 and λ_2 is illustrated in Figure (5.1).

We mentioned at the beginning of Section 3 that I_Λ is invariant under the rotation group $\mathcal{R}$ on Ω. This is also true of the characteristic equation. Hence, if $\Lambda \in \{\Lambda^0\}$, then $\mathcal{R}\Lambda \subset \{\Lambda^0\}$ also. Hence if $\phi_\Lambda^0(\Omega)$ is an eigenfunction, so is any element of $\{\phi_{\mathcal{R}\Lambda}^0(\Omega)\}$.

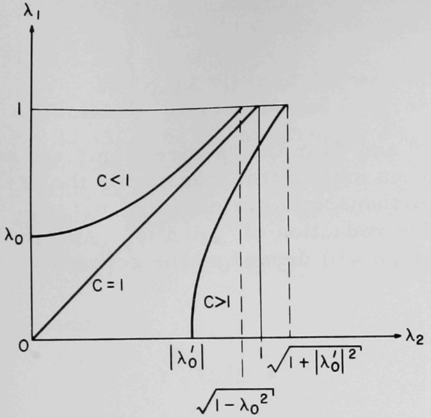

Fig. 5.1

Relationship between λ_1 and λ_2 for $c < 1$, $c = 1$, and $c > 1$

The functions $\psi(r, \Omega)$ that can be represented by the regular eigenfunctions (2.7) are given by the first integral in Eq. (2.15). We select a convenient reference direction from $\{\Lambda^0\}$, say $w^0 + iu^0$, such that, by Eqs. (5.7a) and (5.9c),

$$\left.\begin{aligned} \Lambda_0 &= \Lambda_1^0 + i\Lambda_2^0 \\ &= \sqrt{\lambda_2 + \lambda_0^2}\, w^0 + i\lambda_2 u^0, \end{aligned}\right\} \qquad (5.10)$$

where

$$\left.\begin{aligned} w^0 &= \Lambda_1^0/\lambda_1, \\ u^0 &= \Lambda_2^0/\lambda_2. \end{aligned}\right\} \qquad (5.11)$$

It should be noted that $w^0 \cdot u^0 = 0$ since $\cos\gamma = 0$. An element of $\mathfrak{R}$ is denoted by R. Then

$$\int_{\{\Lambda^0\}} A^0(\Lambda)\,\psi_\Lambda^0 d_4\Lambda = \int_{\lambda_2 = \left\{\begin{array}{l} 0 \text{ for } C \le 1 \\ |\lambda_0| \text{ for } C \ge 1 \end{array}\right\}}^{\lambda_2 = \sqrt{1-\lambda_0^2}} d\lambda_2 \int_{\mathfrak{R}} d_3 R A^0(R\,\Lambda^0)\psi_{R\,\Lambda^0}(\Omega). \qquad (5.12)$$

The order of integration on the right-hand side of this equation may be interchanged.

For applications, it may be convenient to choose for the reference direction an orthonormal system; e.g.,

$$\left.\begin{aligned} u^0 &= \{1, 0, 0\}; \\ v^0 &= \{0, 1, 0\}; \\ w^0 &= \{0, 0, 1\}. \end{aligned}\right\} \qquad (5.13)$$

Then Λ_1^0 is in the "z-direction," and Λ_2^0 in the "x-direction," and an element R is given by the transformation matrix that transforms the orthonormal system (5.13) into another orthonormal system (u, v, w); i.e., R is an orthogonal transformation. The reduction of $\int A(\Lambda^0)\psi^0_{R\Lambda^0}(\Omega)d_3R$ to a triple integral of the Riemannian type will depend on the geometric properties of the problem at hand.

6. GENERALIZED EIGENFUNCTIONS AND THE GENERAL SOLUTION OF THE HOMOGENEOUS EQUATION

In Eq. (2.11) to (2.14) we defined the generalized eigenfunctions. In Section 4, parts B. b) and C. d), we determined the integral term in Eq. (2.13) to be I_Λ, as given by Eq. (4.34). As has been anticipated, G is a function of only λ_1, λ_2, and γ, and does not depend on the particular direction of Ω^0 or the complex vector Λ. It depends only on the relative position of the projections of the real and imaginary parts of Λ into E_3. Hence

$$G = G(\Lambda) = G(\lambda_1, \lambda_2, \gamma) = G(|\Lambda_1|, |\Lambda_2|, \Lambda_1 \Lambda_2), \tag{6.1}$$

or explicitly

$$G = 1 - (c/4\pi)\, I_\Lambda, \tag{6.2}$$

where I_Λ is given by Eq. (4.34) for $\lambda_1 \geq 1$ and $\lambda_1 \sin\gamma \geq 1$.

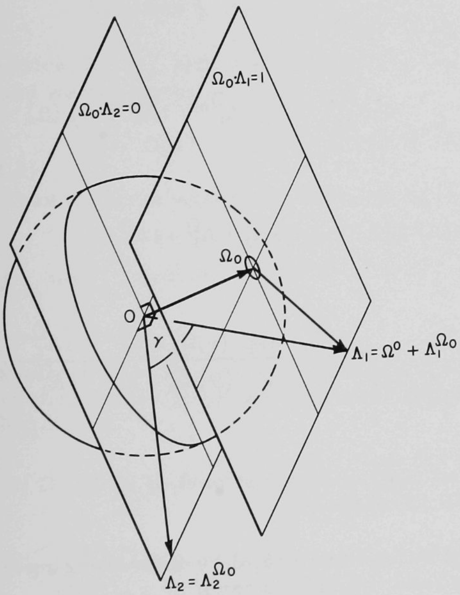

Fig. 6.1. Representation of the Complex Vectors Λ Which Satisfy $\Omega_0 \cdot \Lambda = 1$

To obtain all generalized eigenfunctions which are unbounded at Ω^0, we observe that, by (2.12),

$$1 - \Omega_0 \cdot \Lambda = 0. \tag{6.3}$$

This means that the end points of the real part of the complex vector Λ lie in the plane

$$\Omega_0 \cdot \Lambda_1 = 1, \tag{6.4}$$

which is perpendicular to, and passes through the end point of Ω_0, as illustrated in Figure (6.1). The end points of the imaginary part of Λ lie in the plane

$$\Omega_0 \cdot \Lambda_2 = 0, \tag{6.5}$$

which is perpendicular to Ω_0 and passes through the origin, as illustrated also in Figure (6.1). We see therefore that these vectors Λ satisfy the conditions given in Lemma 3.1 and Corollary (3.2) on pages 12 and 13.

Definition 6.1

All vectors lying in the real plane $\Omega_0 \cdot \Lambda_2 = 0$ shall be denoted by $\{\Lambda^{\Omega_0}\}$.

With this definition, every vector Λ_2 satisfying Eq. (6.5) is defined by

$$\Lambda_2 = \Lambda_2^{\Omega_0}, \tag{6.6}$$

and every vector Λ_1 satisfying (6.4) is defined by

$$\Lambda_1 = \Omega_0 + \Lambda_1^{\Omega_0}, \tag{6.7}$$

since the two planes (6.4) and (6.5) are parallel.

Therefore, the second term in Eq. (2.15) has the representation

$$\int_{\{\Omega_0\}} d_2\Omega_0 \int_{\{\Lambda:\Omega_0\Lambda=1\}} d_4\Lambda\, A(\Lambda)\ \psi_\Lambda(\Omega) =$$

$$\int_{\{\Omega_0\}} d_2\Omega_0 \int_{\{\Lambda^{\Omega_0}\}} d_2\Lambda_1^{\Omega_0} \int_{\{\Lambda^{\Omega_0}\}} d_2\Lambda_2^{\Omega_0}\ A(\Omega_0 + \Lambda_1^{\Omega_0} + i\Lambda_2^{\Omega_0})\ \psi_\Lambda(\Omega) = \tag{6.8}$$

$$\kappa \int_{\{\Lambda^\Omega\}} d_2\Lambda_1^\Omega \int_{\{\Lambda^\Omega\}} d_2\Lambda_2^\Omega\ A(\Lambda)\ G(\Lambda)\ \exp-(r, \Omega + \Lambda_1^\Omega + i\Lambda_2^\Omega) +$$

$$+ \kappa \int_{\{\Omega_0\}} d_2\Omega_0 \int_{\{\Lambda^{\Omega_0}\}} d_2\Lambda_1^{\Omega_0} \int_{\{\Lambda^{\Omega_0}\}} d_2\Lambda_2^{\Omega_0}\ \frac{\exp-(r,\Omega_0 + \Lambda_1^{\Omega_0} + i\Lambda_2^{\Omega_0})}{1 - (\Omega, \Omega_0 + \Lambda_1^{\Omega_0} + i\Lambda_2^{\Omega_0})}\ A(\Lambda), \tag{6.9}$$

where $\Lambda = \Omega + \Lambda_1^\Omega + i\Lambda_2^\Omega$ in $A(\Lambda)$ and $G(\Lambda)$ of the first part of (6.9). G is reduced further by Eq. (6.1) [or Eq. (6.12) below].

The first term in (6.9) was obtained from (6.8) by interchanging Ω and Ω_0 in Definition (2.2) and corresponds to the second term of $\phi_\Lambda(\Omega)$ in Eq. (2.11).

If the integration over Ω^0 is carried out in polar coordinates, we can replace Definition (2.2) by

$$\delta(\Omega - \Omega_0) = \delta(\mu - \mu_0)\, \delta(\phi - \phi_0) \tag{6.10}$$

and thus reduce the two-dimensional δ-function to the product of two one-dimensional δ-functions in a natural way.

In both terms of (6.9), the order of integration with respect to $\Lambda_1^{\Omega_0}$ and $\Lambda_2^{\Omega_0}$ can be interchanged. The actual integration in the $\{\Lambda^{\Omega_0}\}$-plane can be carried out in any way convenient for the specific problem at hand, e.g., in two-dimensional polar coordinates or two-dimensional Cartesian coordinates.

Further simplifications are possible, e.g., as follows: Let us denote the angle between $\Lambda_1^{\Omega_0}$ and $\Lambda_2^{\Omega_0}$ by γ^{Ω_0}, as indicated in Figure (6.2). Now

$$\left.\begin{aligned} \lambda_1 &= \sqrt{1 + (\lambda_1^{\Omega_0})^2}, \\ \Lambda_1 \cdot \Lambda_2 &= (\Lambda_1^{\Omega_0} + \Omega_0,\ \Lambda_2^{\Omega_0}) = \\ \lambda_1 \lambda_2 \cos\gamma &= \lambda_1^{\Omega_0} \lambda_2^{\Omega_0} \cos \gamma^{\Omega_0}. \end{aligned}\right\} \tag{6.11}$$

Hence, λ_1, λ_2, and γ are determined by $\lambda_1^{\Omega_0}$, $\lambda_2^{\Omega_0}$, and γ^{Ω_0}, respectively, and we can write

$$G = G(\Lambda) = G(\lambda_1^{\Omega_0}, \lambda_2^{\Omega_0}, \gamma^{\Omega_0}). \tag{6.12}$$

Thus the first integral in (6.9) can be represented by

$$\int_{\{\Lambda^\Omega\}} d_2\Lambda_1^\Omega \int_{\{\Lambda^\Omega\}} d_2\Lambda_2^\Omega\, A(\Lambda)\, G(\Lambda)\, e^{-(r,\Lambda)} = \int_0^\infty d\lambda_1^\Omega \int_0^\infty d\lambda_2^\Omega \int_0^{2\pi} d\gamma^\Omega\, G \int_{\mathcal{R}\Omega_0} dR^\Omega\, A(R^\Omega \Lambda^I)\, e^{-(r, R^\Omega \Lambda^I)}, \tag{6.13}$$

where R^{Ω_0} is an element of the plane rotation group $\mathcal{R}^{\Omega_0}$, and Λ^I is the reference vector for the rotation such that $\Omega_0 \cdot \Lambda^I = 1$, $\lambda_i^I = \lambda_i^{\Omega_0}$ $(i = 1, 2)$, and $\gamma^I = \gamma^{\Omega_0}$.

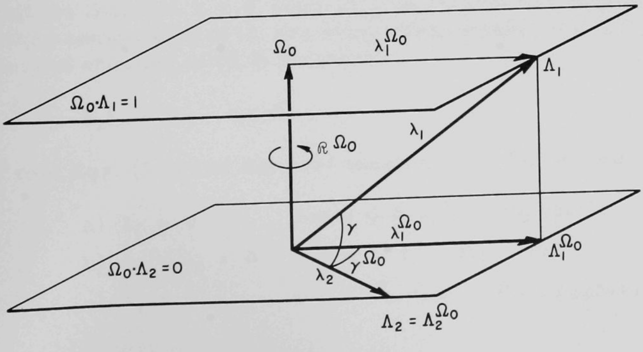

Fig. 6.2

Example of Specific Coordinates for Integration of (6.9)

Hence the most general solution which can be obtained for Eq. (1.4) by superposition of solutions (2.1) is Eq. (2.15), where the integrals appearing in that equation are explicitly given by Eq. (5.12) and (6.9). In manipulating these expressions, one should bear in mind that $G(\Lambda)$ may become zero. The integral expressions for the superposition of the generalized eigenfunctions may be improper integrals. These integrals will be considered as existing if a finite neighborhood of the singularity exists for which the integral converges as the limit of the neighborhood converges to zero. In other words, we admit Cauchy principal-value type integrals over higher dimensions.

7. REDUCTION TO THE KNOWN CASE OF PLANE GEOMETRY

The requirement that $\psi(r,\Omega)$ be a plane solution of Eq. (1.4) means, by definition, that $\psi(r,\Omega)$ be constant for all r whose end points lie on parallel planes. We rotate our coordinate system so that these planes are orthogonal to the x-axis. It follows then, from Eqs. (2.10) and (2.14), that

$$\frac{\partial \psi_\Lambda^0}{\partial y} = \frac{\partial \psi_\Lambda^0}{\partial z} = 0$$

and

$$\frac{\partial \psi_\Lambda}{\partial y} = \frac{\partial \psi_\Lambda}{\partial z} = 0,$$

if and only if

$$\Lambda_y = \Lambda_z = 0. \tag{7.1}$$

Hence the elementary functions have the form

$$\left.\begin{aligned} \psi_\Lambda^0 &= e^{-\Lambda_x x}\phi_\Lambda^0(\Omega), \quad \text{(regular eigenfunctions);} \\ \psi_\Lambda &= e^{-\Lambda_x x}\phi_\Lambda(\Omega), \quad \text{(generalized eigenfunctions).} \end{aligned}\right\} \tag{7.2}$$

We must now submit the regular eigenfunctions to the conditions (5.5) which resulted from the characteristic equation. Because of Eqs. (3.2), (3.3), and (7.1),

$$\left|\mathrm{Re}\Lambda_x\right| = \lambda_1;$$

$$\left|\mathrm{Im}\Lambda_x\right| = \lambda_2.$$

We show that either λ_1 or λ_2 must be zero. From Eq. (7.1) it follows that $\sin\gamma = 0$ since Λ_{1x} is necessarily parallel to Λ_{2x} when all other components of Λ are zero. This means $\cos\gamma = \pm 1$. But then the second equation of (5.5) requires

$$\lambda_1\lambda_2 = 0. \quad \text{q.e.d.}$$

From Eqs. (5.3) and the first equation of (5.5), we deduce that

a) $\lambda_2 = 0$ for $0 < c < 1$, (λ_0 real);

b) $\lambda_1 = \lambda_2 = 0$ for $c = 1$, ($\lambda_0 = 0$);

c) $\lambda_1 = 0$ for $1 < c < \infty$, (λ_0 imaginary).

Hence the regular elementary functions are

$$\psi_{\Lambda}^{0} = e^{\pm\lambda_0 x}\, \phi_{\Lambda}^{0}(\Omega),$$

where

$$\phi_{\Lambda}^{0} = \frac{\kappa c}{4\pi} \cdot \frac{1}{1 \pm \lambda_0 \cdot \Omega_x}.$$

This is in agreement with Reference 1 and consequently also with Reference 2.

Next we must reduce the generalized eigenfunctions to the case of plane geometry.

From Eq. (7.1) it follows, as before, that $\sin\gamma = 0$ since Λ_{1x} and Λ_{2x} are necessarily parallel. But then Eqs. (6.4) and (6.5) cannot be satisfied simultaneously in E_3, unless $\lambda_2 = 0$. This, however, leads to the same results as were obtained in the case of plane geometry as shown by Eq. (4.35). Therefore, the case of plane geometry as already known is contained in our general expansion formula and exhibits a much simpler structure.

8. UNSOLVED PROBLEMS

If we denote by Λ' and Λ'' any two vectors Λ associated with the eigenfunctions ϕ^0_Λ or ϕ_Λ, we obtain from Eq. (2.5) the relation

$$(\Lambda'' - \Lambda') \cdot \int \Omega \phi_{\Lambda'} \phi_{\Lambda''} \, d_2\Omega = 0, \qquad \text{for } \Lambda' \neq \Lambda''. \tag{8.1}$$

In the case of plane geometry, this equation leads to the well-known orthogonality relations given in References 1 and 2. We see at a glance, however, that relation (8.1) is not suited to determine the coefficient functions in the expansion (2.15). Therefore we ask the following unanswered questions:

1) Does a weight function $W(\Omega)$ exist such that

$$\int W(\Omega) \phi_{\Lambda'} \phi_{\Lambda''} \, d_2\Omega = \frac{c\kappa}{4\pi} \int \frac{W(\Omega)}{\Omega \cdot (\Lambda' - \Lambda'')} (\phi_{\Lambda'} - \phi_{\Lambda''}) \, d_2\Omega$$

$$= N(\Lambda', \Lambda'') \, \delta(\Lambda' - \Lambda'') \, ? \tag{8.2}$$

2) How should $\delta(\Lambda' - \Lambda'')$ be defined?

When these questions are answered in a positive sense, we ask further:

3) What function is $W(\Omega)$? How is it determined?

4) What is F in the identities

$$\int_\Omega \left(\int_{\{\Lambda\}} A(\Lambda) \phi_\Lambda d_4\Lambda \right) W(\Omega) \phi_{\Lambda'} \, d_2\Omega \equiv \int_{\{\Lambda\}} A(\Lambda) \left(\int W(\Omega) \phi_\Lambda \phi_{\Lambda'} d_2\Omega \right) d_4\Lambda + F \, ? \tag{8.3}$$

In other words: What is the effect of interchanging the integration with respect to Λ and Ω? One must bear in mind that $\phi_\Lambda \phi'_\Lambda$ are all possible combinations of regular and generalized eigenfunctions. It is therefore likely that we have not only one weight function but at least three different weight functions $W(\Omega)$.

Then arises the question of completeness.

5) In which function space is the set of eigenfunctions as developed in this report complete?

6) Do these eigenfunctions form a minimal base, or is there redundancy?

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